Positioning systems in Minkowski space-time: from emission to inertial coordinates

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Abstract. The coordinate transformation between emission coordinates and inertial coordinates in Minkowski space-time is obtained for arbitrary configurations of the emitters. It appears that a positioning system always generates two different coordinate domains, namely, the front and the back emission coordinate domains. For both domains, the corresponding covariant expression of the transformation is explicitly given in terms of the emitter world-lines. This task requires the notion of orientation of an emitter configuration. The orientation is shown to be computable from the emission coordinates for the users of a 'central' region of the front emission coordinate domain. Other space-time regions associated with the emission coordinates are also outlined.

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1. Introduction

A relativistic positioning system consists of a set of four clocks A ($A = 1, 2, 3, 4$) broadcasting their respective proper times τ^A by means of electromagnetic signals. Then, every event reached by the signals is naturally labelled by the four times $\{\tau^A\}$: the emission coordinates of this event. Elsewhere [1], we have presented a brief report on relativistic positioning and related issues, providing a background with current references on the subject.

A user of a positioning system that receives the four times $\{\tau^A\}$ knows his own coordinates in the emission system. Then, if he wants to know his position in another coordinate system, he must obtain the transformation between both coordinate systems. Thus, we must solve the following important problem in relativistic positioning. Suppose that the world-lines of the emitters $\gamma_A(\tau^A)$ are known in a coordinate system $\{x^\alpha\}$: can the user obtain his coordinates in this system if he knows his emission coordinates $\{\tau^A\}$? Or, slightly more general, can the coordinate transformation $x^\alpha = \kappa^\alpha(\tau^A)$ be obtained?

The main goal of this paper is to solve this question in Minkowski space-time or, more explicitly, to obtain the coordinate transformation between emission and inertial coordinates for arbitrary world-lines of the emitters.

In a two-dimensional approach to relativistic positioning systems this query is rather simple. Indeed, in this case the knowledge of the emitter's world-lines in a chosen null coordinate system gives the coordinate transformation between this null system and the emission coordinates at once [2, 3]. This fact facilitates an analytical study of the positioning systems defined by inertial emitters in Minkowski plane [2], and those defined by stationary emitters in both Minkowski and Schwarzschild planes [3].

The general properties of the emission coordinates for the four-dimensional case have been analyzed in [4]. Nevertheless, in this generic case it is difficult to solve the above stated problem. The study of specific three-dimensional situations [5] has shed light on the complex geometry of the domains and codomains of the emission coordinates. On the other hand, the transformation between emission and inertial coordinates has recently been obtained for emitters following particular inertial motions in flat space-time, and also considering the immediate vicinity of a Fermi observer in the Schwarzschild geometry [6].

Here we present the solution to this problem for a generic configuration of the emitters in Minkowski space-time. We show that two emission domains exist which are called the front emission coordinate domain and the back emission coordinate domain, and we give the coordinate transformation for each one. The transformation $x^\alpha = \kappa^\alpha(\tau^A)$ between inertial and emission coordinates is given in a covariant way in terms of the world-lines $\gamma_A(\tau^A)$ of the emitters. The compact covariant expression of our result is a powerful tool for use in subsequent applications: to obtain the coordinate transformation for specific configurations of the emitters or, under perturbation methods, modeling more realistic gravitational situations.

The paper is structured as follows. In section 2, we pose the problem to be solved, defining the emission regions and the main relations governing the coordinate transformation: the null propagation conditions and the emission conditions. By choosing one of the emitters arbitrarily as reference emitter, in section 3, we show that the null propagation equations of a positioning system are equivalent to a rank-three linear system and a sole quadratic equation. Then, section 4 is devoted to obtaining the general solution of the linear system, and in section 5 we impose the remaining quadratic equation and we obtain a general form for both solutions. In section 6, we apply the emission conditions which guarantee that the solutions are physically interpretable as emission solutions. In section 7 we define the orientation of the positioning system with respect to an event, a concept which allows us to give the main result of this paper in compact form, namely, the explicit expression of the coordinate transformation between emission and inertial coordinates. The computational and/or observational determination of the orientation of the positioning system is analyzed in section 8. Section 9 deals with the analysis of our results and comments about ongoing work in progress and on further practical applications. Finally, an appendix is devoted to

presenting a technical proof of the results given in section 4.

A short communication on this work has been presented at the Spanish Relativity meeting ERE-2008 [7].

2. Statement of the problem. Emission regions and emission relations

2.1. The emission region $\mathcal{R}$ of a positioning system

Let us consider a *positioning system* in Minkowski space-time $\mathcal{M}^4$, and let $\gamma_A(\tau^A)$, $A = 1, \ldots, 4$, be the world-lines of its four distinct emitters A (clocks) watching their proper time τ^A. These four times, broadcast by means of electromagnetic signals, will reach some region, say the *emission region* $\mathcal{R}$, of events of $\mathcal{M}^4$. In the interior of $\mathcal{R}$, the four times $\{\tau^A\}$ converging at every event define generically the so called *emission coordinates* [2, 3, 4, 8, 9].

It is clear that an event P of $\mathcal{M}^4$ belongs to the emission region $\mathcal{R}$ if and only if there exists a past directed null geodesic from P to every emitter $\gamma_A(\tau^A)$ for some value of τ^A. For future discussions, we need to make the ingredients of this assertion more explicit.

Let us denote by x the position vector of P with respect to the origin O of some inertial chart $\{x^\alpha\}$, $x \equiv$ OP, and by γ_A the position vector of the emitters with respect to this chart, $\gamma_A \equiv \mathrm{O}\gamma_A(\tau^A)$. Then, in order for P to belong to $\mathcal{R}$, the four vectors

$$m_A \equiv x - \gamma_A \ , \qquad (1)$$

which represent the trajectories followed by the electromagnetic signals in vacuum issued from the emitters A (see Fig. 1a), must verify the *null propagation conditions* L:

$$\mathsf{L}: \quad (m_A)^2 = 0 \ , \quad \forall A \ .$$

Furthermore, these four vectors have to be future-pointing or, in other words, must verify the *emission conditions* E:

$$\mathsf{E}: \quad \epsilon\, u \cdot m_A < 0 \ , \quad \forall A \ ,$$

where u is any given, everywhere non vanishing, time-like vector field defining the arrow of time and 2ϵ, $\epsilon = \pm 1$, is the metric signature.

The events x where the emission condition E holds must be *receivers*. But it is worth noting that the null propagation conditions L also apply in the case of 'active' events x, able to send or to reflect null signals to one or more of the emitters. Such other *location systems*, as the physical realizations of coordinate systems are called here, have been dealt with elsewhere [10, 11, 12] but will not be considered here.

Nevertheless, in obtaining our coordinate transformations, we will need to consider, besides the emission condition E, its 'causally dual' *reception conditions* R :

$$\mathsf{R}: \qquad \epsilon\, u \cdot m_A > 0 \ , \quad \forall A \ ,$$

both condensed in the *emission-reception conditions* $\mathsf{E\text{-}R}$:

$$\mathsf{E\text{-}R}: \quad \epsilon\, m_A \cdot m_B < 0 \ , \quad \forall A , B \ ,$$

as it is easy to argue.

Now, the above assertion about P and $\mathcal{R}$ may be stated as follows: *an event* P *belongs to the emission region* $\mathcal{R}$ *of a positioning system of emitters* $\gamma_A(\tau^A)$ *if, and only if, its position vector* x *in some inertial chart verifies the null and emission conditions* L *and* E*, respectively, for some values of the proper times* τ^A *of the emitters.*

2.2. The characteristic emission function Θ *of a positioning system and the emission coordinate region* $\mathcal{C}$ *of the space-time*

It then turns out that a positioning system may be considered as a device Θ that physically associates, to every event of the emission region $\mathcal{R}$, a set of four times $\{\tau^A\}$.

From a formal point of view, this device is nothing but an application $\Theta : \mathcal{R} \longrightarrow \mathcal{T}$, henceforth called the *characteristic emission function* of the positioning system, that, with every event x of the emission region $\mathcal{R}$ of the space-time, associates four times τ^A of the *grid* $\mathcal{T}$ of the $\tau's$, $\Theta : x \longmapsto (\tau^A) = \Theta(x)$.

The grid $\mathcal{T}$ is nothing but the Cartesian product $\mathcal{T} \equiv \overset{4}{\times} \{\tau\} \approx \mathbb{R}^4$ of the spaces (real lines) of the variables τ (for details on the concept and role of the grid, see for example [2, 3, 5]).

In this grid $\mathcal{T}$, the image $\Theta(\mathcal{R})$ of the emission region $\mathcal{R}$ by the characteristic emission function Θ, is called the *emission co-region* of the positioning system, and is denoted by ${}^\Theta\!\mathcal{R}$, ${}^\Theta\!\mathcal{R} = \Theta(\mathcal{R})$. The points of this region of the grid $\mathcal{T}$ are the quadruplets of times that can really be received in the space-time, so that they are related to the space-time events at which they are measured. In this sense, this emission co-region ${}^\Theta\!\mathcal{R}$ is the sole region of the grid $\mathcal{T}$ which possesses a physical meaning (the other quadruplets of $\mathcal{T}$ are a convenient mathematical completion of the emission co-region but with *no* relation to space-time events).

Obviously, if the world-lines of the four emitters are sufficiently smooth and broadcast their proper time continuously, the emission co-region ${}^\Theta\!\mathcal{R} \subset \mathcal{T}$ is connected and, because of the regularity of the light cones in Minkowski space-time, $\mathcal{R}$ is connected too. But this property does not guarantee that the emission function Θ, $\tau^A = \Theta^A(x)$, is invertible in $\mathcal{R}$.

For Θ to be invertible, the gradients $d\tau^A$, normal to the hypersurfaces $\tau^A =$ constant, have to be *well defined* and *linearly independent*. But, because in Minkowski space-time our light cones are everywhere differentiable up to on their vertices, i.e. on the emitter world-lines γ_A, the gradients $d\tau^A$ are well defined everywhere in $\mathcal{R}$ up to on $\mathcal{R} \cap \big(\cup_A \gamma_A(\tau^A)\big)$. Then, on the region $\mathcal{R} - \big(\cup_A \gamma_A(\tau^A)\big)$, because the $d\tau^A$ are metrically collinear to the above null vectors m_A, the linear independence of the $d\tau^A$ may be expressed by the *coordinate condition* C:

$$\mathsf{C}: \quad m_1 \wedge m_2 \wedge m_3 \wedge m_4 \neq 0 \ .$$

This condition C is equivalent to say that $j_\Theta(x) \neq 0$, where $j_\Theta(x)$ is the determinant of the Jacobian matrix $J_\Theta(x)$ of Θ, so that the locus where the $d\tau^A$ are linearly dependent

is the hypersurface $\mathcal{J}$ in $\mathcal{R}$ of equation $\mathcal{J} \equiv \{j_\Theta(x) = 0\}$. We shall call the regions $\mathcal{D} \equiv \mathcal{J} \cup (\cup_A \gamma_A(\tau^A))$ and $\mathcal{C} \equiv \mathcal{R} - \mathcal{D}$ the *emission degenerate region*, and the *emission coordinate region* respectively, $\mathcal{R} = \mathcal{C} \cup \mathcal{D}$.

It is to be noted that, the condition C being satisfied on its events, $\mathcal{C}$ is an open set. For the same reason, Θ is invertible in $\mathcal{C}$, so that all its events may be locally labeled by the coordinates $\{\tau^A\}$. Nevertheless, this does not means necessarily that $\mathcal{C}$ be a *coordinate domain* of a local chart $(\mathcal{C}, \Theta)$, because the condition C assures only the *local* invertibility of Θ. In fact, we will see that $\mathcal{C}$ is *not* a coordinate domain, but the union of *two* coordinate domains; this is why we have called $\mathcal{C}$ the emission coordinate *region*. The region in the grid $\mathcal{T}$ where the characteristic emission function Θ, $\tau^A = \Theta^A(x)$, is locally invertible is the region ${}^\Theta\mathcal{C} \equiv \Theta(\mathcal{C})$, and will be called the *emission coordinate co-region* of the grid $\mathcal{T}$.

2.3. The main relations of a positioning system

When the world-lines of the emitters are known, the *main relations* of a positioning system are the set $\{\mathsf{L}, \mathsf{E}\}$ of the null propagation conditions L and the emission conditions E.

In terms of the position vectors γ_A of these world-lines $\gamma_A(\tau^A)$, they give rise, by (1), to the *null propagation equations*

$$\mathsf{L}: \quad (x - \gamma_A) \cdot (x - \gamma_A) = 0 \ , \quad \forall A \ . \tag{2}$$

and to the *emission inequalities*

$$\mathsf{E}: \quad \epsilon\, u \cdot (x - \gamma_A) < 0 \ , \quad \forall A \ , \tag{3}$$

where u is any given, everywhere non vanishing, future-pointing time-like vector.

To invert the function Θ in the emission coordinate co-region ${}^\Theta\mathcal{C}$ is to solve the main relations (2), (3) in x, $x = \kappa(\tau^A)$, for values of the τ^A's verifying the coordinate condition C or, by (1), the *coordinate inequality*:

$$\mathsf{C}: \quad (x - \gamma_1) \wedge (x - \gamma_2) \wedge (x - \gamma_3) \wedge (x - \gamma_4) \neq 0 \ . \tag{4}$$

These solutions $x = \kappa(\tau^A)$ may alternatively be read as the position vectors of the events whose past light cone cuts the emitter world-lines $\gamma_A(\tau^A)$ at their times τ^A. The components $x^\alpha = \kappa^\alpha(\tau^A)$ of x then define the coordinate transformation between the emission coordinates $\{\tau^A\}$ and the inertial ones $\{x^\alpha\}$.

The main object of this paper is to obtain this coordinate transformation.

3. The null propagation equations L

When $\{\tau^A\}$ are the emission coordinates of the event x, the emitters are at the events $\{\gamma_A(\tau^A)\}$. These four events define the internal *configuration* of the emitters for the event x.

One can try to solve the null propagation equations (2) in x, $x = \kappa(\tau^A)$ directly, but it is better to first carefully separate the ingredients intrinsically related to the

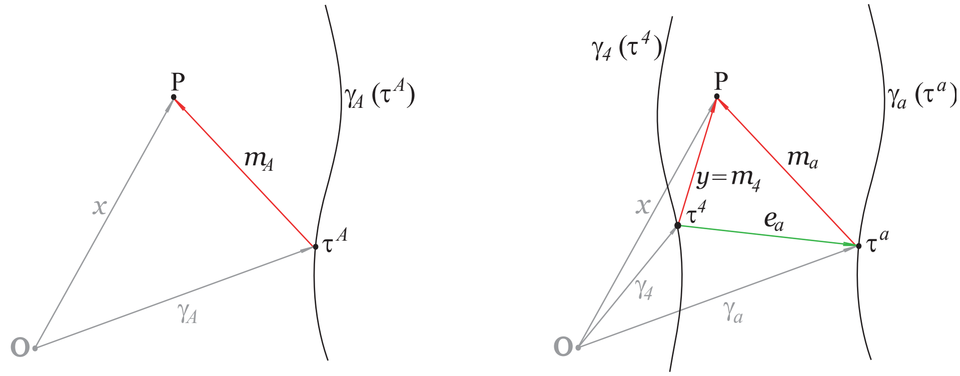

Figure 1. (a) The light-like vectors $m_A \equiv x - \gamma_A$ represent the trajectories followed by the electromagnetic signals between the emitters and an event P in the emission region. x is the position vector of this event with respect to some inertial chart, and γ_A, $A = 1, 2, 3, 4$, are the position vectors of the emitters with respect to this chart. (b) If we choose the emitter 4 as origin (reference emitter), the relative positions of the others (referred emitters) are $e_a = \gamma_a - \gamma_4$, $a = 1, 2, 3$, and the position vector of the event P is $y = m_4$.

configuration of the four emitters A, which are independent of the inertial chart $\{x^\alpha\}$, from the ingredients related to this chart, which is independent of the configuration of the emitters.

For this purpose, we shall arbitrarily choose one of the emitters, say $A = 4$, as *reference emitter* and we shall search the solution x under the form:

$$x = \gamma_4 + y \ , \tag{5}$$

where y is the solution to the null propagation equations (2) when the origin is chosen at the position γ_4 of the emitter 4 (see Fig. 1b). Thus, y is submitted to

$$\mathsf{L}_4 : \quad (y - e_A) \cdot (y - e_A) = 0 \ , \quad \forall A \ , \tag{6}$$

where

$$e_A = \gamma_A - \gamma_4 \ . \tag{7}$$

Now, splitting these relations for $A = 4$ and $A = a = 1, 2, 3$, we have

$$e_4 = 0 \ , \quad e_A = e_a \tag{8}$$

and, for the null propagation equations (6),

$$\begin{array}{ll} y^2 = 0 \ , & A = 4 \ , \\ y^2 - 2\, e_a \cdot y + (e_a)^2 = 0 \ , & A = a \ . \end{array} \tag{9}$$

Noting that e_a are the relative position vectors of the *referred* emitter $A = a$ with respect to the reference emitter $A = 4$, and that the half of their squares, represent the *world functions* [13] of the referred and reference emitters, we have:

Lemma 1 *The null propagation equations* (6) *are equivalent to the* main linear system

$$e_a \cdot y = \Omega_a \ , \quad a = 1, 2, 3 \ , \tag{10}$$

and the main quadratic equation

$$y^2 = 0 \ , \tag{11}$$

where the scalars Ω_a, world functions of the endpoints of the vectors e_a, are given by

$$\Omega_a \equiv \frac{1}{2}(e_a)^2 \ . \tag{12}$$

In the study of the null propagation equations we are supposing that the emitted times $\{\tau^A\}$ received at x constitute the emission coordinates of x so that the coordinate inequality (4) fulfils. From the definitions (7) this condition writes

$$\mathsf{C}_4: \quad \chi \cdot y \neq 0 \ , \tag{13}$$

where χ is the *configuration vector* defined by

$$\chi \equiv *(e_1 \wedge e_2 \wedge e_3) \ , \tag{14}$$

with $*$ being the Hodge dual operator. Note that, under the form C_4 of C, χ is necessarily non vanishing:

$$\chi \neq 0 \ , \tag{15}$$

which means that the configuration $\{\gamma_A(\tau^A)\}$ determines a hyperplane. This *configuration hyperplane* contains the four events $\{\gamma_A(\tau^A)\}$ and is orthogonal to the configuration vector χ. The sign of χ^2 provides the causal character (space-like, light-like or time-like) of this hyperplane. Thus, in the emission coordinate region $\mathcal{C}$ one can distinguish three disjoint regions:

the *space-like configuration region*	$\mathcal{C}_s \equiv \{x \in \mathcal{C} \mid \epsilon\, \chi^2 < 0\}$,
the *null configuration region*	$\mathcal{C}_\ell \equiv \{x \in \mathcal{C} \mid \ \chi^2 = 0\}$,
the *time-like configuration region*	$\mathcal{C}_t \equiv \{x \in \mathcal{C} \mid \epsilon\, \chi^2 > 0\}$.

with $\mathcal{C} = \mathcal{C}_s \cup \mathcal{C}_\ell \cup \mathcal{C}_t$.

The ingredients Ω_a and χ of the internal configuration of the emitters make use of the relative position vectors e_a and their double exterior product $e_1 \wedge e_2 \wedge e_3$. Let us complete them by introducing their simple exterior products in the form

$$E^a \equiv *(e_{a+1} \wedge e_{a+2}) \ , \tag{16}$$

where the operations on the indices obviously have to be understood modulo 3.

In looking for the solutions to the null propagation equations (6), it is convenient to first start solving the main linear system (10) in y and then constraint it to verify the main quadratic equation (11). Then, the solution(s) $y = \iota(\tau^A)$ so obtained, incremented by the position γ_4 of the reference emitter $A = 4$, will give by (5) the solution(s) to the null propagation equations (2), $x = \kappa(\tau^A) = \gamma_4 + \iota(\tau^A)$, i.e. the wanted coordinate transformation between emission coordinates and inertial ones. The next section is devoted to solving the main linear system (10).

4. The main linear system

The regularity condition (15) for the configuration vector χ means that the rank of the main linear system (10) is exactly 3. As a consequence, its general solution y depends on a sole parameter, say λ, so that denoting by y_* a particular solution, one has $y = y_* + \lambda\chi$ because, by (14), χ is a vector such that $e_a \cdot \chi = 0$. We can thus state:

Lemma 2 *In the emission coordinate region $\mathcal{C}$, the general solution to the main linear system* (10) *is of the form*

$$y = y_* + \lambda\chi \ , \tag{17}$$

where the parameter λ takes arbitrary values, χ is the configuration vector (14) *and y_* is a particular solution.*

A tentative study of explicit expressions of the particular solution y_* shows that, for every emission coordinate domain, *there is not a sole analytical function, depending exclusively of the elements of the configuration $\{\gamma_A(\tau^A)\}$ of the emitters, that is valid throughout the domain.* This is why it is *necessary* to introduce an external element to obtain a sole analytical expression. We shall see that it is *sufficient* for this external element to be a vector field ξ *transversal* to the configuration, i.e. such that

$$\xi \cdot \chi \neq 0 \ , \tag{18}$$

and otherwise arbitrary. Denoting by $i(\ )$ the interior product, it may then be shown (see Appendix A for the proof):

Proposition 1 *In all the emission coordinate region $\mathcal{C}$, the general solution to the main linear system* (10),

$$e_a \cdot y = \Omega_a \ , \quad a = 1, 2, 3 \ ,$$

is of the form

$$y = y_* + \lambda\chi \ , \tag{19}$$

where the parameter λ takes arbitrary values, χ is the configuration vector

$$\chi \equiv *(e_1 \wedge e_2 \wedge e_3) \ ,$$

and y_* is the particular solution orthogonal to a chosen transversal vector ξ, $\xi \cdot \chi \neq 0$, given by

$$y_* \equiv \frac{1}{\xi \cdot \chi} \ i(\xi)H \ , \tag{20}$$

where the bivector H is, like the vector χ, a function of the configuration of the emitters,

$$H \equiv \Omega_a E^a \ , \quad E^a \equiv *(e_{a+1} \wedge e_{a+2}) \ . \tag{21}$$

The direct and simple discussion that has enabled us to reach proposition 1 is the result of a careful search in order to find, in every coordinate domain, a *sole*, *general* and *covariant* expression for the coordinate transformation between emission coordinates $\{\tau^A\}$ and inertial ones $\{x^\alpha\}$.

The last two requirements of generality and covariance enable us to find *once and for all* and *for any inertial system* (and in *compact form* due to the intrinsic vector formalism) the coordinate transformation in question. But they do not suffice to lead by themselves to a sole expression valid for all the configurations, i.e. valid at *all* the events of every emission coordinate domain. For example, if we were to work directly on the main linear system in the cases of a configuration generating a nonnull vector χ, we would find the solution

$$y = \frac{1}{\chi^2}\, i(\chi)H + \lambda\chi \ , \tag{22}$$

as now directly follows from (20) and (19) for the choice $\xi = \chi$. But, meanwhile in fact there is neither physical nor mathematical reason for the solution to lose analyticity, this expression (22) becomes manifestly undefined on the null configurations, $\chi^2 = 0$. This is why it is *necessary* to introduce an external element to find the sole regularized analytical expression above mentioned.

This external element, the transversal vector field ξ, independent of the configuration of the emitters, is otherwise arbitrary. But, meanwhile due to lemma 2 we know that the main linear system depends on a sole parameter λ, the expressions (19), (20) of y apparently seem to indicate that it depends on the $(4 + 1 =)$ 5 arbitrary parameters $\{\xi; \lambda\}$. Thus, what is the precise role played by this transversal field ξ?

From the definition (21) of the bivector H it is clear that its dual $*H$ is orthogonal to χ, $i(\chi) * H = 0$, so that, one has $H = \chi \wedge a$ for some vector a. With this expression of H it is easy to prove from (19) and (20) that the variation of the solution y with respect to the parameters ξ verifies

$$\frac{\partial y}{\partial \xi} \wedge \chi = 0 \ , \tag{23}$$

which shows that the changes in ξ may be absorbed by λ, i.e. that the parameters ξ^α are not *essential* in the Eisenhart sense [14]. Nevertheless their elimination gives rise to different expressions for the causally different regions that one can find in the whole emission coordinate region.

5. The main quadratic equation

Obtaining the transformation $x^\alpha = \kappa^\alpha(\tau^A)$ between emission coordinates $\{\tau^A\}$ and inertial ones $\{x^\alpha\}$ amounts to determine the intersection of the four future light cones emitted by the emitters at the configuration $\{\gamma_A(\tau^A)\}$ or, in the dual interpretation, to determine the vertex of the past light cone that contains the emitters in such a configuration.

According to proposition 1, the solutions y of the main linear system (10) depend linearly of the parameter λ, and consequently, describe a straight line in $\mathcal{M}^4$. The vertex that we are searching for is consequently on this straight line and, according to our choice of reference emitter, is the point on it related to the emitter $A = 4$ by the null vector $m_4 = x - \gamma_4 = y$. This is nothing more than what the main quadratic equation (11) expresses: $y^2 = 0$. This cone, whose vertex is the emitter $A = 4$, cuts the straight line (19) at a value of λ given implicitly by:

$$\chi^2\lambda^2 + 2(\chi \cdot y_*)\lambda + y_*^2 = 0 \,, \tag{24}$$

and it is this value of λ which, put in the expression (19) of proposition 1, enables us to obtain the coordinate transformation $y = \iota(\tau^A)$ that we are searching for.

But, before obtaining it, and in order to be sure of its real character, it is convenient to observe that the sole assumption we have made so far to solve the null propagation equations is that of the regularity of the configuration data $\{\gamma_A(\tau^A)\}$, i.e. of the non-vanishing of the configuration vector χ, $\chi \neq 0$. This assumption is only a *strict* part of the coordinate condition C, as its form C_4, $\chi \cdot y \neq 0$, shows.

The study of degenerate configuration data $\chi = 0$ and of the degenerate coordinate condition $\chi \cdot y = 0$ is interesting to gain more in-depth knowledge of the physical conditions leading to their presence, in order to avoid, control or predict them. But we shall restrict ourselves here, as already stated, to obtaining the coordinate transformation between emission coordinates $\{\tau^A\}$ and inertial ones $\{x^\alpha\}$. For this purpose, from now on, we shall work under the coordinate condition C.

In fact, it is this condition that guarantees the real character of the solutions of the null propagation system. To demonstrate this, we shall separately analyze the cases of null and non null configurations.

Null configurations. In the region where the configuration vector is null, $\chi^2 = 0$, expression (19) for the solution y, $y = y_* + \lambda\chi$, leads to $\chi \cdot y = \chi \cdot y_*$, so that the last expression (13) for the coordinate condition C ensures the well defined character of λ in (24),

$$\lambda = -\frac{y_*^2}{2(\chi \cdot y_*)} \,, \tag{25}$$

and we have

Proposition 2 *Under the coordinate condition* C*, for null configurations,* $\chi^2 = 0$*, the null propagation equations* (6) *admit a real and single solution* y *given by:*

$$y = y_* - \frac{y_*^2}{2(\chi \cdot y_*)}\,\chi \,, \tag{26}$$

where y_* *and* χ *are respectively given by* (20) *and* (14)*.*

Non-null configurations. In the regions where the configuration vector is non-null, $\chi^2 \neq 0$, (24) gives:

$$\lambda_\pm = \frac{1}{\chi^2}\left(-\chi \cdot y_* \pm \sqrt{\Delta}\right) \,, \quad \Delta \equiv (\chi \cdot y_*)^2 - \chi^2\, y_*^2 \,. \tag{27}$$

But, for these non-null configurations, we can choose $\xi = \chi$ in the expression (20) of the particular solution y_* to the main linear system (10). Let us denote by c the corresponding y_* :

$$c = \frac{1}{\chi^2}\, i(\chi)H\,. \tag{28}$$

Because H is antisymmetric, it is obvious that c is orthogonal to χ, so that, for the corresponding value λ_c of λ, (27) takes the simple form:

$$\lambda_{c\pm} = \pm\frac{\sqrt{\Delta_c}}{\chi^2}\ , \quad \Delta_c \equiv -\chi^2\, c^2\ . \tag{29}$$

Now, expression (19) for the solution y, $y = c + \lambda_c\,\chi$, leads, by product by χ, to $y\cdot\chi = \lambda_c\,\chi^2$. Taking its square and substituting in it the value (29) of λ_c, one obtains:

$$(y\cdot\chi)^2 = \Delta_c \quad , \tag{30}$$

so that the form (13) of the coordinate condition C ensures the strict positiveness of the discriminant Δ_c. Then, for $y_* = c$ and the values (29) of λ_c, equations (19) lead, after a little analysis, to the following result:

Proposition 3 *Under the coordinate condition* C*, for non-null configurations* $\chi^2 \neq 0$*, the null propagation equations* (6) *admit two real solutions* $y_\pm$ *given by*

$$y_\pm = c \pm |c|\,\nu\,, \qquad \nu \equiv \frac{\chi^2}{|\chi|^3}\chi\,, \tag{31}$$

where c *and* χ *are, respectively, given by* (28) *and* (14)*.*

In the above proposition and in what follows we denote $|v|$ the modulus of a vector v, $|v| \equiv \sqrt{|v^2|}$.

We have seen that, for the particular solution $y_* = c$, the discriminant Δ given by (27) takes the value Δ_c given by (29). To what extent the discriminant Δ, corresponding to other particular solutions $y_* \neq c$, differs from the value Δ_c?

The question is pertinent because we want to unify expressions (26) and (31) so as to have a single expression for every coordinate domain, irrespective of the causal orientation of the configuration vector χ. To answer this, note that c is the particular solution to the linear system orthogonal to χ and that, from (19), any y_* differs from c by a term of the form $\lambda\chi$. More precisely, $c = y_* - \frac{y_*\cdot\chi}{\chi^2}\chi$. Then, a straightforward calculation leads to:

Lemma 3 *Under the coordinate condition* C*, for non-null configurations* $\chi^2 \neq 0$*, the discriminant* Δ *and each one of the solutions* $y_\pm$ *of the quadratic equation* (24)*, given in* (27)*, are independent of the particular solution chosen and, therefore, of the subsidiary vector* ξ *too. The discriminant is positive and its invariant value is*

$$\Delta = \Delta_c = -\chi^2 c^2 > 0 \quad , \tag{32}$$

where c *is given by* (28)*.*

Unified expression for any configuration. As already stated, we want to find, for every coordinate domain, a *sole* expression valid for the *whole* domain, i.e. valid for *all* the configurations $\{\gamma_A(\tau^A)\}$ that could correspond to the events of the domain. Nevertheless, in principle, by continuously changing of events in the coordinate domain, we can make these configurations to have a vanishing, positive or negative value of χ^2. When this is the case, *none* of the two *standard expressions* (27) of λ for $\chi^2 \neq 0$ comes down to the sole solution (25) for $\chi^2 = 0$ and, consequently, none of the expressions (31) reduces to expression (26), both of them becoming degenerate. To correct this degeneration, it is sufficient to multiply numerator and denominator of the second member of (27) by the conjugate of the numerator. Once this is done, taking into account proposition 1 one has the following result.

Theorem 1 *In all the emission coordinate region $\mathcal{C}$, the solutions to the null propagation equations* (6) *are real and admit the expression:*

$$y_\pm = y_* + \lambda_\pm \chi\,, \quad \lambda_\pm \equiv -\frac{y_*^2}{(\chi \cdot y_*) \pm \sqrt{\Delta}}\,, \quad \Delta \equiv (\chi \cdot y_*)^2 - \chi^2\, y_*^2\,, \tag{33}$$

where y_* and χ are respectively given by (20) *and* (14). *For null configurations $\chi^2 = 0$, the sole solution is given by one of these expressions, the other becoming degenerate.*

6. The emission conditions E

The expressions (33) of theorem 1 give the Minkowski events y that can be related to the configuration events $\{\gamma_A(\tau^A)\}$ by means of emission or reception of null signals. But we want y to be the events reached by null signals *emitted* from the configuration events. Thus we have to impose on y the emission conditions E. This will select a special class of configurations, which will be called *emission configurations*.

In a first step we shall impose the intermediate emission-reception condition E-R, guaranteeing that either all the signals at y have been emitted by the configuration, or all the signals at y will be received by the configuration; such a configuration will be called an *emission-reception configuration*. And in a second step we shall choose with E the first of these two cases.

As we have seen in section 2, the emission-reception condition E-R states that all the null vectors m_A joining γ_A and y, $m_A \equiv y - \gamma_A$, must have the same (past or future) orientation, i.e. $\epsilon\, m_A \cdot m_B < 0\,,\ \forall\, A, B$. In terms of the relative position vectors e_a of the configuration $\{\gamma_A(\tau^A)\}$ given by (7) and (8), one has:

$$\begin{aligned} m_4 - m_a &= \gamma_a - \gamma_4 = e_a \\ m_a - m_b &= \gamma_b - \gamma_a = e_b - e_a \quad , \end{aligned} \tag{34}$$

so that, because $(m_A - m_B)^2 = -2\, m_A \cdot m_B$, we have:

Proposition 4 *For a regular configuration $\{\gamma_A(\tau^A)\}$ to be an emission-reception configuration it is necessary and sufficient that all their relative positions be space-like.*

In other words: the null directions m_A verify the emission-reception condition E-R *if, and only iff,*

$$(\mathsf{E\text{-}R})_4 : \quad \epsilon\,(e_a)^2 > 0 \quad , \quad \epsilon\,(e_a - e_b)^2 > 0 \quad . \tag{35}$$

Let us note that, in particular, this proposition tells us that, in the grid $\mathcal{T} \approx \mathbb{R}^4$ of the τ's, the domains of emission coordinates $\{\tau^A\}$ are in the interior of the region determined by (35), so that the points of the grid $\mathcal{T}$ in the complementary region certainly have *no* physical meaning.

Equations (33) of theorem 1 show us that, under the coordinate condition C, two real and definite solutions to the null propagation system may correspond to every configuration $\{\gamma_A(\tau^A)\}$. Now, under the additional emission-reception condition E-R, one of these definite solutions may be that of emission and the other of reception. This last one finally has to be detected and discarded by means of the emission condition E, which, under the E-R one, reduces to:

$$\mathsf{E}_4 : \quad \epsilon\, y \cdot u < 0\,, \tag{36}$$

where u is any given, everywhere non vanishing, future-pointing time-like vector.

For non-null configurations, $\chi^2 \neq 0$, the general expression (33) gives the two admissible solutions under the coordinate condition C. These solutions also admit the non-unified expression given in proposition 3. From it we obtain:

$$y_+ \cdot y_- = 2\,c^2 \tag{37}$$

Then, (32) implies that if $\epsilon\,\chi^2 > 0$ (*respectively*, $\epsilon\,\chi^2 < 0$) then $\epsilon\,c^2 < 0$ (*respectively*, $\epsilon\,c^2 > 0$) and, as a consequence of (37), y_+ is future-pointing iff y_- is future-pointing (*respectively*, past-pointing). Thus, we have:

Lemma 4 *If χ is a time-like vector, $\epsilon\,\chi^2 < 0$, then only one of the solutions* (33) *corresponds to an emission configuration.*

If χ is a space-like vector, $\epsilon\,\chi^2 > 0$, then the two solutions of (33) *correspond to either two emission configurations or two reception configurations.*

When χ is a time-like vector, can we detect the solution which corresponds to an emission configuration? The answer is affirmative. Indeed, as a consequence of lemma 3, each one of the solutions $y_\pm$ is independent of ξ. Thus, taking for them the non-unified expression given in proposition 3, we obtain (when $\epsilon\,\chi^2 < 0$):

$$\epsilon\,(\epsilon\chi) \cdot y_\pm = \chi \cdot y_\pm = \pm |c|\,|\chi|\,,$$

and, consequently, we can state:

Proposition 5 *For a space-like emission configuration, $\epsilon\,\chi^2 < 0$, the sole emission solution is y_+ (*respectively*, y_-) if $\epsilon\chi$ is past-pointing (*respectively*, future-pointing).*

When χ is a space-like vector, we can know a priori whether the two solutions $y_\pm$ correspond to emission configurations. Indeed, in this case c is a time-like vector. Then, making use again of the non-unified expression for $y_\pm$ given in proposition 3, we obtain:

$$\epsilon\, c \cdot y_\pm = \epsilon c^2 < 0\,,$$

and, consequently, we can state:

Proposition 6 *For a time-like emission configuration, $\epsilon\,\chi^2 > 0$ the solutions $y_\pm$ correspond to emission (*respectively*, reception) configurations if c is future-pointing (*respectively*, past-pointing).*

Finally, for a null configuration, $\chi^2 = 0$, proposition 2 gives the sole admissible solution under the coordinate condition C, which of course also reduces to one of the solutions of the general expression (33). In this case $\Delta = (y_* \cdot \chi)^2$ and, consequently, the non degenerate solution in (33) is y_+ (*respectively*, y_-) if $\epsilon\,(\epsilon\chi) \cdot y_* = \chi \cdot y_* > 0$ (*respectively*, < 0). Thus, we can state:

Proposition 7 *For a null emission configuration, $\chi^2 = 0$, the non degenerate solution is y_+ (*respectively*, y_-) if $\epsilon\chi$ is past-pointing (*respectively*, future-pointing).*

It is worth remarking the attachment between the results stated in propositions 5 and 7. If we continuously change to a null configuration coming from a space-like one, the configuration vector χ keeps its future or past orientation. If χ is past-pointing (*respectively*, future-pointing) we must take the (continuous) expression y_+ (*respectively*, y_-) in both the null and space-like regions.

7. The coordinate transformation from emission to inertial coordinates

7.1. Front and back emission coordinate domains

Theorem 1 shows that the emission coordinate region $\mathcal{C}$ is mapped with *two* local charts and gives analytical expressions for the transformation between emission and inertial coordinates for the two corresponding coordinate domains.

These expressions of the coordinate transformation and the study of the emission conditions in section 6 show that the causal character of the configuration of the emitters differs for the two emission coordinate domains. Space-like and null configurations only admit one of the solutions given in theorem 1 as emission solution. And, of course, the coordinate domain of this solution contains by continuity a time-like configuration region. The coordinate domain of the other solution only contains a time-like configuration region.

We see that the two coordinate domains differ enough in the causal character of their emitter configurations. We shall call *front emission coordinate domain* $\mathcal{C}^F$ the coordinate domain that contains events with the three possible causal configurations. More precisely, $\mathcal{C}^F$ contains all the space-like configuration region $\mathcal{C}_s$, all the null configuration region $\mathcal{C}_\ell$ and a part $\mathcal{C}_t^F$ of the time-like configuration region $\mathcal{C}_t$. We shall call *back emission coordinate domain* $\mathcal{C}^B$ the other coordinate domain, that only contains events with time-like configurations.‡ In fact, $\mathcal{C}^B = \mathcal{C}_t - \mathcal{C}_t^F$.

‡ Let us remember that it is usual to call *coordinate domain* the open set U of any local chart (U, ϕ) of the atlas defining a differentiable manifold. This appellation requires attention because a coordinate domain is not necessarily a domain, but simply a (not necessarily connected) topological open set. In

Moreover, the two time-like regions have the same codomain in the grid, namely, $\Theta(\mathcal{C}_t^F) = \Theta(\mathcal{C}^B)$. This relation is very important. It says that, whatever be the four values $\{\tau^A\}$ received by a user in the coordinate region $\mathcal{C}_t^F$ (resp. $\mathcal{C}^B$), another user in the coordinate region $\mathcal{C}^B$ (resp. $\mathcal{C}_t^F$) may receive the *same* values $\{\tau^A\}$. In other words: a user that only receives four times $\{\tau^A\}$ defining a time-like configuration of the emitters is *unable* to detect in what part of the region $\mathcal{C}_t$, in $\mathcal{C}_t^F$ or in $\mathcal{C}^B$, he is.

7.2. Orientation of a positioning system

To be able to detect in what of these regions a user is, a notion of orientation of a positioning system is necessary. We give the following one:

Definition 1 *The* orientation *of a positioning system with respect to an event of its emission coordinate region $\mathcal{C}$ is the sign $\hat{\epsilon}$ of the coordinate condition scalar:*

$$\hat{\epsilon} \equiv sgn * (m_1 \wedge m_2 \wedge m_3 \wedge m_4) \quad . \tag{38}$$

We have seen in section 2.2 that, in the emission region $\mathcal{R}$, the hypersurface $\mathcal{J}$ separates the emission coordinate region $\mathcal{C}$ in two open sets. We can now identify them with the above front and back coordinate domains $\mathcal{C}^F$ and $\mathcal{C}^B$, respectively. Because $\mathcal{J}$ is the hypersurface where the coordinate condition C is not verified (vanishing Jacobian $j_\Theta(x) = 0$), the non vanishing member of its inequality has a constant sign in every coordinate domain $\mathcal{C}^F$ and $\mathcal{C}^B$, so that we have the simple but important result:

Proposition 8 *The orientation $\hat{\epsilon}$ of a positioning system is constant in every one of the coordinate domains $\mathcal{C}^F$ and $\mathcal{C}^B$.*

The same way that leads to the form (13) of the coordinate condition C shows that $\hat{\epsilon} = sgn\,(y \cdot \chi)$. Because y is necessarily future-pointing for a emitted signal, if χ is time-like or null we can be sure that the sign of $(y \cdot \chi)$ is the same as that of the sign of $(u \cdot \chi)$ for any everywhere non vanishing future-pointing time-like vector u. This last sign is plus or minus according to the past- or future-pointing character of $\epsilon\chi$. Taking into account propositions 5 and 7, one has the following:

Proposition 9 *In the regions $\mathcal{C}_s$ and $\mathcal{C}_\ell$, the orientation $\hat{\epsilon}$ of a positioning system is given by*

$$\hat{\epsilon} = sgn\,(u \cdot \chi) \tag{39}$$

for any future-pointing time-like vector u.

fact, meanwhile the front emission coordinate domain $\mathcal{C}^F$ is generically connected, the back emission coordinate domain $\mathcal{C}^B$ is generically the disjoint union of four connected components [5].

7.3. Explicit expression of the coordinate transformation from emission to inertial coordinates

We shall comment below in section 8 that for users in the region $\mathcal{C}_t$ the determination of the orientation needs additional information. But for the moment, Theorem 1 and the above propositions lead us to the following result.

Theorem 2 *Let γ_A be the position vectors of the world-line equations $\gamma_A(\tau^A)$ of the four emitters of a positioning system with respect to an inertial coordinate system $\{x^\alpha\}$, and $\{\tau^A\}$ their emission coordinates. In all the emission coordinate region $\mathcal{C} = \mathcal{C}^F \cup \mathcal{C}^B$, the coordinate transformation $x = \kappa(\tau^A)$ is given by:*

$$x = \gamma_4 + y_* - \frac{y_*^2\,\chi}{(y_* \cdot \chi) + \hat{\epsilon}\sqrt{(y_* \cdot \chi)^2 - y_*^2\chi^2}} \tag{40}$$

where y_* is the quantity given by (20)*, χ is the configuration vector* (14) *and $\hat{\epsilon}$ is the orientation* (38) *of the positioning system with respect to the event that receives the data $\{\tau^A\}$.*

This is the main result reached in this paper. It rest to analyze now to what extent this expression (40) may be determined by a user from the data received by him.

8. The central region of a positioning system. Computational and observational orientation

The world-lines $\gamma_A(\tau^A)$ of the emitters in an inertial system $\{x^\alpha\}$, as well as the space-time metric in it, are here supposed known 'background' data for any user of the positioning system (the world-lines can be pre-determined initially or broadcast in real time). Thus, any user who receives only the emission data $\{\tau^A\}$ is able to compute the quantities γ_4, y_*, χ appearing in (40). He has to follow the four steps:

Step 1. Compute the four position vectors of the emitters, $\gamma_A = \mathrm{O}\gamma_A(\tau^A)$, for the received values $\{\tau^A\}$.

Step 2. Choose a reference emitter, say γ_4, and compute the position vector of the three referred emitters $e_a = \gamma_a(\tau^a) - \gamma_4(\tau^4)$.

Step 3. Compute the configuration scalars $\Omega_a \equiv \frac{1}{2}e_a^2$, the configuration vector $\chi \equiv *(e_1 \wedge e_2 \wedge e_3)$, and the configuration bivectors $E^a \equiv *(e_{a+1} \wedge e_{a+2})$ and $H \equiv \Omega_a E^a$.

Step 4. Choose a transversal vector ξ, $\xi \cdot \chi \neq 0$, and compute $y_* \equiv \frac{1}{\xi \cdot \chi} i(\xi)H$.

At this level, the user has computed the quantities γ_4, y_*, χ. But he is also able to compute in what of the coordinate regions, $\mathcal{C}_t$, $\mathcal{C}_\ell$ or $\mathcal{C}_s$ of the positioning system he is; for this, one additional step is sufficient:

Step 5. Determine the sign of $\epsilon\chi^2$.

Then, according to their definitions, the user is in $\mathcal{C}_t$, $\mathcal{C}_\ell$ or $\mathcal{C}_s$ if this sign of $\epsilon\chi^2$ is > 0, $= 0$ or < 0 respectively.

Now, suppose that the user is in $\mathcal{C}_\ell$ or $\mathcal{C}_s$. Another step allows him to compute the orientation $\hat{\epsilon}$:

Step 6. Determine the sign of $u\cdot\chi$ for a future-pointing time-like vector u, arbitrarily chosen.
Then, according to proposition 9, $\hat{\epsilon} = sgn\,(u \cdot \chi)$.

We shall call *central region* of a positioning system the region $\mathcal{C}^C \equiv \mathcal{C}_s \cup \mathcal{C}_\ell$. A part of theorem 2 may be then stated as follows.

Proposition 10 *Let γ_A be the position vectors of the known world-line equations $\gamma_A(\tau^A)$ of the four emitters with respect to an inertial coordinate system. Then the users of the central region $\mathcal{C}^C$ of the positioning system can obtain their position $\{x^\alpha\}$ in the inertial system by* computation *from their sole emission coordinates $\{\tau^A\}$.*

What about the users in $\mathcal{C}_t$, i.e. out of the central region $\mathcal{C}^C$? We have seen that everywhere $\hat{\epsilon} = sgn\ (y \cdot \chi)$. Nevertheless, the null vector y involves not only the initial data $\gamma_A(\tau^A)$ and the received data τ^A, but is the solution we are looking for, so that the quantity $y \cdot \chi$ cannot be computed before itself. For this reason proposition 9 is *exclusive*: *only* the users of the central region $\mathcal{C}^C$ are able to *compute* the orientation $\hat{\epsilon}$ of the positioning system.

Can the users in $\mathcal{C}_t$ know the orientation of the positioning system? It is possible to show that all the users of the coordinate region $\mathcal{C}$, be them in the central region $\mathcal{C}^C$ or not, can determine the orientation of the positioning system if, in addition to the reception of the $\{\tau^A\}$, they are able to *observe* the emitters in their celestial sphere. But this fact will be analyzed in a forthcoming paper.

9. Discussion and work in progress

In this paper we have obtained the coordinate transformation (40) between emission coordinates and inertial coordinates in Minkowski space-time. We use the intrinsic vector formalism to express the position vector, $x \equiv (x^\alpha)$, of every event in the emitter coordinate region as a function f of the emitter world-lines $\gamma_A(\tau^A) : x = f(\mathrm{O}\gamma_A(\tau^A)) \equiv \kappa(\tau^A)$.

This general and compact expression of the coordinate transformation will be a powerful tool for subsequent applications. For example, we can particularize it for different choices of the emitter world-lines which model specific physical situations. In doing so, a previous basic task appears to be convenient in many cases: to write our covariant expressions in a 3+1 formalism with respect to an arbitrary inertial observer, which is the goal of another work [15]. Moreover, from the expression of the coordinate transformation we can easily obtain the components of the metric tensor in emission coordinates and we can study the region where the emission coordinates are more efficient than the inertial ones [16].

In section 6 we have studied the emission conditions in order to distinguish between emission and reception or mixed configurations. These results are a necessary tool to carry out more in-depth analysis of the domains and co-domains of the emission coordinates. This task will be tackled in another paper [17] where we will also study

the geometry of the emitter configurations attending to their different causal character, and we will analyze how this geometry influences the solutions to the null propagation equations. Some preliminary results on this question have been presented in [7].

The orientation $\hat{\epsilon}$ of a positioning system with respect to an event (see section 7) is a concept which has allowed us to give an explicit expression of the coordinate transformation in theorem 2. In section 8 we have pointed out that this orientation $\hat{\epsilon}$ may be computed from the emission data $\{\tau^A\}$ in the so-called central region. Nevertheless, out of this region, the determination of $\hat{\epsilon}$ demands an observational method which will be analyzed elsewhere.

The development of all this theoretical work paves the way to the study of more realistic, gravitationally influenced, positioning systems. For example, those defined by emitter world-lines modeling a satellite constellation around the Earth in a weak gravitational field. It is worth remarking that for the use and smooth running of a positioning system one needs, not only the coordinate transformation to emission coordinates but also a good understanding of the domains in the space-time and co-domains in the grid. The analysis of the degenerate configuration data, $\chi = 0$, and of the hypersurfaces of the emission region $\mathcal{R}$ where the Jacobian $j_\Theta(x)$ vanishes must also be well understood. We already know that the events of vanishing Jacobian are, and only are, those for which any user in them can see the four emitters on a circle in his celestial sphere [5]. This includes the possibility for the user seeing less than four satellites, when some of them are in the shadows of the others.

Obviously, in realistic situations, the inertial coordinate system considered here must give rise to more useful ones. Namely: the International Celestial Reference System (ICRS) for the positioning system based on four millisecond pulsars, valid for the Solar System, as proposed in [18]; the Barycentric Celestial Reference System (BCRS), to compare the planet trajectories (Earth at least and already) obtained in this millisecond pulsar system with the ones obtained by standard astronomical observations; the Geocentric Celestial Reference System (GCRS) or related World Geodetic System 84 (WGS84) or International Earth Reference System (ITRS) for the applications of relativistic positioning systems to the Global Navigation Satellite Systems (GNSS).

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Appendix A. Proof of proposition 1

From lemma 2, to prove proposition 1 we must obtain a particular solution of the main linear system (10). More precisely, we must obtain the particular solution y_* which is orthogonal to a chosen transversal vector ξ, $\xi \cdot \chi \neq 0$.

In order to obtain such a solution in a covariant way we begin by studying a regular linear system $\phi_A \cdot z = \Omega_A$. In this case, the vectors $\{\phi_A\}$ define a base, that is, $\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4 \neq 0$. Then, the sole solution to the system takes the expression $z = \Omega_A \, \phi^A$, $\{\phi^A\}$ being the dual base. More explicitly, we have:

Lemma 5 *The solution z to the regular linear system*

$$\phi_A \cdot z = \Omega_A \; , \qquad A = 1, 2, 3, 4 \; , \tag{A.1}$$

is given by

$$z = \Omega_A \, \phi^A \; . \tag{A.2}$$

where

$$\phi^A \equiv \frac{1}{3!D} \epsilon^{APQR} * (\phi_P \wedge \phi_Q \wedge \phi_R) \; , \quad D \equiv *(\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4) \; . \tag{A.3}$$

Now, we regularize the main linear system (10) by adding a new equation as follows. Given a vector ξ such that $\xi \cdot \chi \neq 0$, let us consider a linear system of the form (A.1) for the unknown z, with

$$\phi_A = e_a \; , \qquad \Omega_A = \Omega_a \; , \quad A = 1, 2, 3 \; ; \tag{A.4}$$

$$\phi_4 = \xi \; , \qquad \Omega_4 = 0 \; . \tag{A.5}$$

This linear system is regular since:

$$\begin{aligned} D &\equiv *(\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4) = *(e_1 \wedge e_2 \wedge e_3 \wedge \xi) \\ &= - * (\xi \wedge e_1 \wedge e_2 \wedge e_3) = -i(\xi) * (e_1 \wedge e_2 \wedge e_3) = -\xi \cdot \chi \neq 0 \; . \end{aligned} \tag{A.6}$$

Consequently, the sole solution to this system may be obtained as stated in lemma 5. Now $\Omega_4 = 0$, and then we only need to calculate the vectors ϕ_a, $a = 1, 2, 3$, of the dual basis given in (A.3). They take, in this case, the expression:

$$\begin{aligned} \phi^a &\equiv \tfrac{1}{3!D} \epsilon^{aPQR} * (\phi_P \wedge \phi_Q \wedge \phi_R) = \tfrac{1}{2\,D} \epsilon^{abc4} * (e_b \wedge e_c \wedge \xi) \\ &= \tfrac{1}{2\,D} \epsilon^{abc} * (\xi \wedge e_b \wedge e_c) = -\tfrac{1}{2\,D} \epsilon^{abc} i(\xi) * (e_b \wedge e_c) = \tfrac{1}{\xi \cdot \chi} i(\xi) E^a \; , \end{aligned} \tag{A.7}$$

where

$$E^a \equiv *(e_{a+1} \wedge e_{a+2}) \; . \tag{A.8}$$

Then, the solution (A.2) becomes now

$$z = \frac{1}{\xi \cdot \chi} \; i(\xi) H \; , \qquad H \equiv \Omega_a E^a \; . \tag{A.9}$$

Finally, note that this vector z is a solution to the main linear system and it is orthogonal to ξ. Thus it is the particular solution y_* that we are looking for.

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